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# Hilbert–Schmidt separability probabilities and noninformativity of priors

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## Abstract

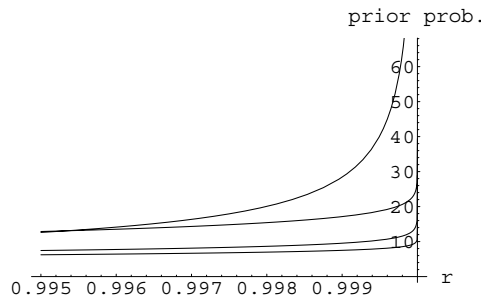
The Horodecki family employed the Jaynes maximum-entropy principle, fitting the mean ( $b_1$ ) of the Bell–CHSH observable ( $B$ ). This model was extended by Rajagopal by incorporating the dispersion ( $\sigma_1^2$ ) of the observable, and by Canosa and Rossignoli, by generalizing the observable ( $B_\alpha$ ). We further extend the Horodecki *one*-parameter model in *both* these manners, obtaining a *three*-parameter ( $b_1, \sigma_1^2, \alpha$ ) two-qubit model, for which we find a highly interesting/intricate continuum ( $-\infty < \alpha < \infty$ ) of Hilbert–Schmidt (HS) separability probabilities—in which, the *golden ratio* is featured. Our model can be contrasted with the three-parameter ( $b_q, \sigma_q^2, q$ ) one of Abe and Rajagopal, which employs a  $q$ (Tsallis)-parameter rather than  $\alpha$ , and has simply  $q$ -invariant HS separability probabilities of  $\frac{1}{2}$ . Our results emerge in a study initially focused on embedding certain information metrics over the two-level quantum systems into a  $q$ -framework. We find evidence, in this regard, that Srednicki’s recently-stated biasedness criterion for *noninformative* priors yields rankings of priors fully consistent with an information-theoretic test of Clarke, previously applied to quantum systems by Slater.

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(Some figures in this article are in colour only in the electronic version)

## 1. Introduction

Both Rajagopal [1], as well as Canosa and Rossignoli [2] have extended a well-known maximum-entropy model of the Horodecki family [3] to *two*-parameter models, but in different fashions. Rajagopal incorporated the *dispersion* ( $\sigma_1^2$ ) of the Bell–CHSH observable ( $B$ ) [4], the *mean* ( $b_1$ ) of which is already fitted in the Horodecki model, while Canosa and Rossignoli fitted the mean of *generalized* Bell–CHSH observables ( $B_\alpha$ ). We combine their two approaches into a *three*-parameter ( $b_1, \sigma_1^2, \alpha$ ) model, for which we uncover a very interesting continuum ( $-\infty < \alpha < \infty$ ) of exact Hilbert–Schmidt separability probabilities (section 7.2.1, figure 11, (37))—in which, among other features, the *golden ratio* [5, 6] appears.



**Figure 1.** Four univariate marginal prior probability distributions in the near-to-pure-state region  $r \in [1 - \epsilon, 1]$ , where  $r$  is the radial coordinate in the Bloch sphere representation of two-level quantum systems, and  $r = 1$  corresponds to a pure state. The order of dominance *fully* complies with that (1) obtained by the information-theoretic-based comparative noninformativity test.

Our model can be interestingly contrasted with a three-parameter  $(b_q, \sigma_q^2, q)$  one also of Abe and Rajagopal [7] (section 7.2), which incorporates the  $q$ -parameter (nonextensivity/Tsallis index/escort parameter), rather than the  $\alpha$ -parameter of  $B_\alpha$  (34). The continuum (over  $q$ ) of separability probabilities (*independently* of the metric employed) is simply a constant (equal to  $\frac{1}{2}$  in the Hilbert–Schmidt case, and to the ‘silver mean’,  $\sqrt{2} - 1$ , for the Bures and other monotone metrics). We examine a *two*-parameter  $(b_1, \alpha)$  Canosa–Rossignoli-type model also exhibiting an interesting (non-flat) continuum (section 7.2.2, figure 12, (38)). Additionally, we obtain exact (Hilbert–Schmidt and Bures) separability probabilities for the three-parameter Tsallis–Lloyd–Baranger model [8] (section 7.3).

Our results emerge in a study initially focused on embedding certain information metrics over the two-level quantum systems into a  $q$ -framework. We find evidence (section 5, figure 1) that Srednicki’s recently-stated biasedness criterion for *noninformative* priors [9] yields rankings of priors fully consistent with an information-theoretic test of Clarke [10], previously applied to quantum systems by Slater [11].

## 2. Noninformativity of priors

Some 15 years ago, Wootters asserted that ‘there does not seem to be any natural measure on the set of all mixed states’ [12, p 1375]. He did, however, consider random density matrices with all eigenvalues *fixed*. He remarked that once ‘the eigenvalues are fixed, then all the density matrices in the ensemble are related to each other by the unitary group, so it is natural to use the unique unitarily invariant measure to define the ensemble’ [12, p 1375] (cf [13]).

Arguing somewhat similarly, Srednicki recently proposed that in choosing a prior distribution over density matrices, ‘we can use the principle of indifference, applied to the unitary symmetry of Hilbert space, to reduce the problem to one of choosing a probability distribution for the eigenvalues of  $\rho$ . There is, however, no compelling rationale for any particular choice; in particular, we must decide how biased we are towards pure states’ [9, p 6].

To be specific, we find, in an analysis involving four prior probabilities ( $p$ ’s), that the *information-theoretic*-based comparative noninformativity test devised by Clarke yields a ranking

$$P_{F_{q=1}} > P_B > P_{B_{q=1}\text{trunc}} > P_F \quad (1)$$

fully consistent (figure 1) with Srednicki’s recently-stated criterion for priors of ‘biasedness to pure states’. Two of the priors are formed by *extending* certain metrics of quantum-theoretic interest from three to four dimensions—by incorporating the  $q$ -parameter. The three-dimensional metrics are the Bures (minimal monotone) metric over the two-level quantum systems and the Fisher information metric over the corresponding family of Husimi distributions. The priors  $p_B$  and  $p_F$  are the (*independent-of- $q$* ) normalized volume elements of these metrics, and  $p_{F_{q=1}}$  is the normalized volume element of the  $q$ -*extended* Fisher information metric, with  $q$  set to 1. While originally intended to similarly be the normalized volume element of the  $q$ -extended Bures metric, with  $q$  then set to 1, the prior  $p_{B_{q=1}\text{trunc}}$  actually entails the *truncation* of the only *off*-diagonal entry of the extended Bures metric tensor. Without this truncation, the  $q$ -extended Bures volume element is *null*, as is also the case in three *higher*-dimensional quantum scenarios we examine (including the Abe–Rajagopal two-qubit states (section 7.2), for which we further find  $q$ -*invariant* Bures separability probabilities equal to  $\sqrt{2} - 1$  (the ‘silver mean’) and Hilbert–Schmidt ones equal to  $\frac{1}{2}$  (section 7.2), and the Tsallis–Lloyd–Baranger two-qubit states (section 7.3)).

### 3. Bures metric

The Bures (minimal monotone) metric—the volume element of which we normalize to obtain one ( $p_B$ ) of the four prior probability distributions of principal interest here—yields the statistical distance between neighbouring mixed quantum states ( $\rho$ ) [14, 15]. It provides an embedding of the Fubini–Study metric [16, section IV], which gives the statistical distance between neighbouring pure quantum states ( $|\psi\rangle$ ) (cf [17]). Hübner gave an explicit formula for the Bures distance [18, p 240] (cf [19]),

$$d_B(\rho_1, \rho_2)^2 = 2 - 2 \operatorname{tr}(\rho_1^{1/2} \rho_2 \rho_1^{1/2})^{1/2}. \tag{2}$$

Further, he expressed it in an infinitesimal form as [18, equation (10)]

$$d_B(\rho, \rho + d\rho)^2 = \sum_{ij} \frac{1}{2} \frac{| \langle i | d\rho | j \rangle |^2}{\lambda_i + \lambda_j}, \tag{3}$$

where the  $\lambda_i$ ’s are the eigenvalues and the  $|i\rangle$ ’s, the eigenvectors of  $\rho$ .

#### 3.1. Three-dimensional case

In [20], using the familiar Bloch sphere (unit ball in Euclidean 3-space) representation of the two-level quantum systems ( $2 \times 2$  density matrices),

$$\rho = \frac{1}{2} \begin{pmatrix} 1+z & x-iy \\ x+iy & 1-z \end{pmatrix}, \quad r^2 = x^2 + y^2 + z^2 \leq 1, \tag{4}$$

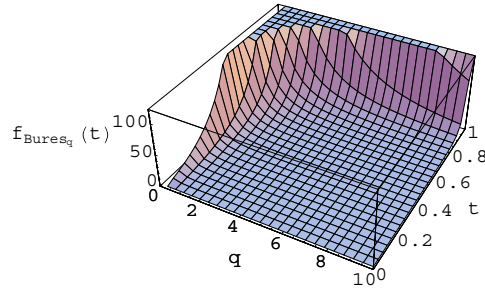
it was found (cf [13, p 128]), here converting from cartesian to spherical coordinates,

$$x = r \cos \theta_1, \quad y = r \sin \theta_1 \cos \theta_2, \quad z = r \sin \theta_1 \sin \theta_2, \tag{5}$$

that

$$d_B(\rho, \rho + d\rho)^2 = \frac{1}{4} \left( \frac{1}{(1-r^2)} dr^2 + dn^2 \right). \tag{6}$$

The term  $dr^2$  corresponds to the radial component of the metric and  $dn^2$ , the tangential component ( $dn^2 = r^2 d\theta_1^2 + r^2 \sin^2 \theta_2 d\theta_2^2$ ). In the setting of the quantum *monotone* metrics—the Bures metric serving as the *minimal* monotone one—it is appropriate to express the



**Figure 2.** The function  $f_{\text{Bures}_q}(t)$  that yields the *tangential* component (11) of the extended (four-dimensional) Bures metric (10).

tangential component of the Bures metric (6) in the form [16, equation (3.17)],

$$\left( (1+r)f_B \left( \frac{1-r}{1+r} \right) \right)^{-1}, \tag{7}$$

where  $f_B(t) = \frac{1+t}{2}$  is an *operator monotone* function [21].

The volume element of the Bures metric (7) is  $\frac{r^2 \sin \theta_1}{8(1-r^2)}$ , which can be normalized to a *prior* probability distribution over the Bloch sphere,

$$p_B = \frac{r^2 \sin \theta_1}{\pi^2(1-r^2)}. \tag{8}$$

### 3.2. Four-Dimensional case

Now, we can construct a *four-dimensional* family of (properly normalized/unit trace)  $2 \times 2$  *escort* density matrices (cf [22]),

$$\rho_{\{q\}} = ((1-r)^q + (1+r)^q)^{-1} \begin{pmatrix} 1+z & x-iy \\ x+iy & 1-z \end{pmatrix}^q, \tag{9}$$

for which  $q = 1$  recovers the standard Bloch sphere representation (4). Applying Hübner’s formula (3), we have found that the *extended* Bures metric (now incorporating the  $q$ -parameter) has the form

$$d_{\text{Bures}_q}(\rho, \rho + d\rho)^2 = \frac{1}{4(1+W^q)^2} \left( W^q \log^2 W dq^2 + \frac{4q W^q \log W}{r^2 - 1} dq dr + 4 \frac{q^2 W^q}{(r^2 - 1)^2} dr^2 + \frac{(-1 + W^q)^2}{r^2} dn^2 \right), \tag{10}$$

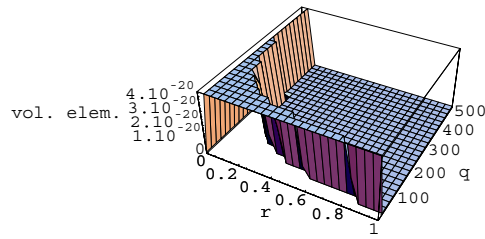
where  $W = \frac{1-r}{1+r}$ , that is, the ratio of the two eigenvalues of  $\rho$ .

The *tangential* component of the metric (10) can be expressed as  $((1+r)f_{\text{Bures}_q}(W))^{-1}$ , where

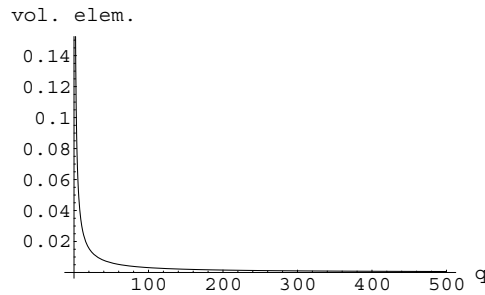
$$f_{\text{Bures}_q}(t) = \frac{2(1+t)(1+t^q)^2}{(-1+t^q)^2}. \tag{11}$$

This bivariate function appears (figure 2) to be monotonically increasing for any fixed  $q$  (cf [16]).

Now, in the earlier stage of our analyses, due to a programming oversight, we were under the impression that the off-diagonal  $dq dr$  term of (10) was simply zero. If we do employ



**Figure 3.** Two-dimensional marginal of the *truncated* four-dimensional extended Bures volume element (10).



**Figure 4.** One-dimensional marginal (12) over  $q$  of the four-dimensional *truncated* extended Bures volume element (10).

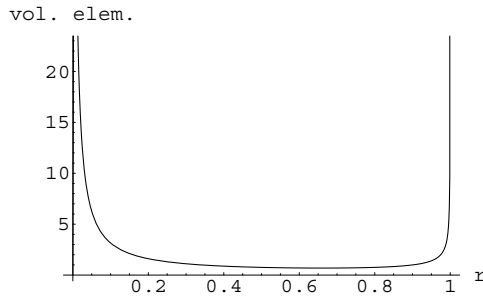
the fully correct form, with this  $dq dr$  term included, we find that the volume element is *null*. This, of course, could not yield a meaningful prior probability distribution. However, having proceeded under the impression that the  $dq dr$  term was null, we obtained a number of results that appear to be of interest and of some relevance. Therefore, for much of this study, we will treat the  $dq dr$  term as null, and thus deal with a *truncated*  $q$ -extended Bures metric.

In the context of the harmonic oscillators states, Pennini and Plastino have argued that, in addition to the physical lower bound (ignorance-amount) of  $q \geq 0$  that in a quantal regime,  $q$  can be no *less* than 1 [23]—due to the Lieb bound on the Wehrl entropy [24]. However, for the two-level quantum systems to the study of which we restrict ourselves here, the lower bound on the Wehrl entropy is  $\frac{1}{2}$  [25, equation (12)]. We, thus, consider  $q \in [\frac{1}{2}, \infty]$  to be the range of possible values of the escort parameter  $q$ . In practice, though, we will, for numerical and graphical purposes and normalization of the (divergent over  $q \in [1/2, \infty]$ ) truncated extended Bures volume element (section 6), consider that  $q \in [\frac{1}{2}, 500]$ .

In figure 3 we show the *two-dimensional marginal* volume element of (10) (after omission of the  $dq dr$  term)—integrating out the spherical angles,  $\theta_1, \theta_2$ , and leaving the radial coordinate  $r$  and the escort parameter  $q$ . In figure 4, further integrating out  $r$ , we show the corresponding *one-dimensional marginal* volume element of (10) (after omission of the  $dq dr$  term) over  $q$ . This (figure 4) has the exact expression

$$\frac{\pi(1 + \log 4)}{24q}. \tag{12}$$

This prior, thus, conforms to Jeffreys’ rule—as opposed to the Bayes–Laplace rule, which would give a *constant* prior [26].



**Figure 5.** One-dimensional marginal (obtained from (13)) over  $r$  of the four-dimensional extended Bures volume element (10) after omission of the off-diagonal  $dq dr$  term.

In figure 5, we integrate out  $q \in [\frac{1}{2}, 500]$ , leaving a (deep bowl-shaped) one-dimensional marginal over  $r \in [0, 1]$ . (The corresponding marginal in the unextended Bures case is  $\frac{\pi r^2}{2(1-r^2)}$ , so it is simply increasing with  $r$ , in that case.) The associated indefinite integral over  $q$  is

$$\frac{\pi(qW^q(3+W^{2q})\log(W) - (1+W^q)(2W^q + (1+W^q)^2\log(1+W^q)))}{6(-1+r^2)(1+W^q)^3\log(W)}. \tag{13}$$

(So, we obtain the function plotted in figure 5 by substituting  $q = 500$  and  $q = \frac{1}{2}$  into (13) and taking the difference.)

For  $q = 1$ , the *extended* Bures metric (10) reduces to

$$ds_{\text{Bures}_{q=1}}(\rho, \rho + d\rho)^2 = \frac{1}{16}(1-r^2)\log^2 W dq^2 - \frac{1}{4}\log W dq dr + ds_B(\rho, \rho + d\rho)^2. \tag{14}$$

Normalizing the volume element of this metric—but first nullifying the off-diagonal  $dq dr$  term—to a (non-null) prior probability distribution over the Bloch sphere, we obtain (cf (8))

$$p_{B_{q=1}\text{trunc}} = \frac{3 r^2 \sin \theta_1 \log \frac{1}{W}}{4 \pi (1 + \log 4)}, \tag{15}$$

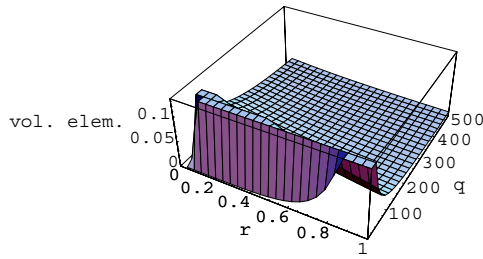
one of the four priors that we rank (figure 1 and (1)) both by the comparative noninformativity test and Srednicki’s biasedness criterion.

### 3.3. Comparative noninformativities in the Bures setting

The *relative entropy* (Kullback–Leibler distance/information gain [27, 28]) of  $p_B$  with respect to  $p_{B_{q=1}\text{trunc}}$  (which we denote  $S_{\text{KL}}(p_B, p_{B_{q=1}\text{trunc}})$ )—that is, the *expected* value with respect to  $p_B$  of  $\log \frac{p_B}{p_{B_{q=1}\text{trunc}}}$ —is 0.101 846 ‘nats’ of information. Now, reversing arguments,  $S_{\text{KL}}(p_{B_{q=1}\text{trunc}}, p_B) = 0.066 1775$ . (We use the *natural* logarithm, and not 2 as a base, with one nat equalling 0.531 bits.) Let us convert—using Bayes’ rule—these two (prior) probability distributions to *posterior* probability distributions ( $post_B$  and  $post_{\text{Bures}_{q=1}}$ ), by assuming *three* pairs of spin measurements, *one* each in the  $x$ -,  $y$ - and  $z$ -directions, each pair yielding one ‘up’ and one ‘down’. This gives us the *likelihood* function (cf [9, equation (9)] [29, equation (4.2)]),

$$L(x, y, z) = \frac{(1-x^2)(1-y^2)(1-z^2)}{64} \tag{16}$$

(which we convert to the spherical coordinates (5) in which we perform our Mathematica computations).



**Figure 6.** Two-dimensional marginal of the four-dimensional extended Husimi volume element (21).

Then, we have  $S_{\text{KL}}(\text{post}_B \| p_{B_{q=1}\text{trunc}}) = 0.169\,782$  and  $S_{\text{KL}}(\text{post}_{\text{Bures}_{q=1}} \| p_B) = 0.197\,657$ . The relative magnitudes of the information gains obtained by passing from priors to posteriors (0.101 846 to 0.169 782 and 0.066 1775 to 0.197 657) seems to suggest that  $p_B$  is somewhat *more* noninformative than  $p_{B_{q=1}\text{trunc}}$ . This is confirmed, using the testing structure given in [11, 30] (cf [9]), if we *formally* use a likelihood  $(L(x, y, z))^{\frac{1}{2}}$ , which is the square root of (16), to compute  $\text{post}_B$  and  $\text{post}_{\text{Bures}_{q=1}}$ . Then, we see a *decrease* in relative entropy from 0.101 846 to 0.093 849 and an *increase* from 0.066 1775 to 0.114 669. So,  $p_B$  can be made *closer* to  $p_{B_{q=1}\text{trunc}}$  by *adding* information to it, but not *vice versa*, leading us to conclude that  $p_B$  is *more* noninformative than  $p_{B_{q=1}\text{trunc}}$ , since it assumes *less* about the data. (Let us note, however, that in the class of monotone metrics [16], the Bures or minimal monotone metric appears to be the *least* noninformative (cf [13, section 5]). The *maximal* monotone metric, on the other hand, is *not* normalizable to a proper prior probability distribution over the Bloch sphere [11]. So, there is an interesting question of whether there exists a *single*, distinguished *normalizable* monotone metric which is *maximally* noninformative.)

#### 4. Fisher information metric of Husimi distributions

Let us now move to a classical context, employing the (generalized) Husimi distributions [31], rather than density matrices to represent the two-level quantum systems. Use of the Fisher information (monotone) metric [32, 33] is now indicated. To generate the (properly normalized) *escort* Husimi distributions  $(H_{\{q\}})$  (cf [23]), from the Husimi distribution  $(H = H_{\{1\}})$ , we employ the formula (cf (9)),

$$H_{\{q\}} = 2(r + qr)(-(1 - r)^{1+q} + (1 + r)^{1+q})^{-1} H^q. \tag{17}$$

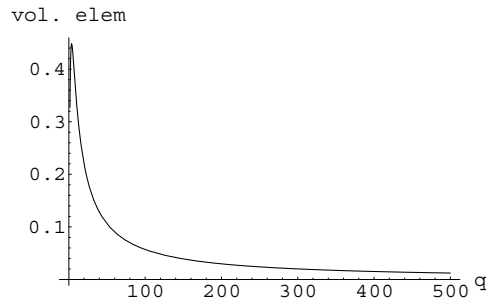
The tangential components of the Fisher information metric for the escort Husimi distributions  $(H_{\{q\}})$  are of the form  $((1 + r)f_{F_q}(t))^{-1}$ , where [30, equation (29)]

$$f_{F_q}(t) = \frac{(-1 + q)(-1 + t)^2(-1 + t^{1+q})}{q(1 + t)(1 - q + t + qt - t^q - qt^q - t^{1+q} + qt^{1+q})}. \tag{18}$$

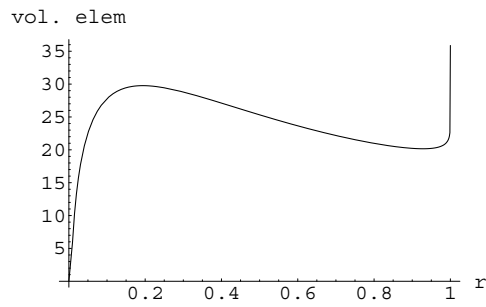
In [30, section V D], we succeeded in finding similarly general (for all  $q$ ) formulae for the denominators, but not the numerators, of the radial components.

In figure 6 we show (having to resort to some *numerical* integrations, since we lack explicit  $[q\text{-general}]$  expressions for certain of the metric elements) the counterpart to figure 3 for the four-dimensional extended *Husimi* metric. Continuing with our numerical methods, we obtain the interesting unimodal curve (figure 7)—the peak being near  $q = 3.597\,82$ , with a value





**Figure 7.** One-dimensional marginal over  $q$  of the four-dimensional extended Husimi volume element (21). There is a peak near  $q = 3.59782$ .



**Figure 8.** One-dimensional marginal over  $r$  of the four-dimensional extended Husimi volume element (21). The upturn near  $r = 1$  may be due to (hard-to-avoid) numerical inaccuracy.

there of 0.448 488. This portrays the one-dimensional marginal Husimi volume element over  $q$  (cf figure 4). In figure 8 we show the (quite difficult-to-compute) one-dimensional marginal over  $r$  (cf figure 5). (It appears the upturn near  $r = 1$  may be simply a numerical artefact. The difficulty consists in that, in some sense, we have to repeatedly perform numerical integrations using results of other numerical integrations. It would be of interest to see how the curve changes as the range of  $q \in [\frac{1}{2}, 500]$  is modified.)

4.1. Three-dimensional metric

For the case  $q = 1$ , the (unextended) three-dimensional Fisher information metric over the family of Husimi distributions takes the form [30, equation (2)]

$$ds_F(\rho, \rho + d\rho)^2 = \frac{-2r - \log\left(\frac{1-r}{1+r}\right)}{2r^3} dr^2 + \left( (1+r)f_F\left(\frac{1-r}{1+r}\right) \right)^{-1} dn^2. \tag{19}$$

Here,

$$f_F(t) = \frac{(t-1)^3}{t^2 - 2t \log t - 1}, \tag{20}$$

which is the limiting case ( $q \rightarrow 1$ ) of (11). To normalize the volume element of this metric (19) to a prior probability distribution ( $p_F$ ), we divide it by 1.393 509 89 [30].

#### 4.2. Four-dimensional metric

In the extended (four-dimensional) case (cf (14)), after having set  $q = 1$ , we have

$$ds_{F_{q=1}}(\rho, \rho + d\rho)^2 = \left( \frac{1}{4} - \frac{(-1 + r^2)^2 \log^2 W}{16r^2} \right) dq^2 + \frac{2r - (-1 + r^2) \log W}{2r^2} dq dr + ds_F(\rho, \rho + d\rho)^2. \quad (21)$$

(So, the metric tensor here, in the same manner as in the *untruncated* extended Bures case (10), is not fully diagonal. We do *not* truncate the  $q$ -extended Fisher information metric (21) in any of our analyses.) To normalize its (non-null) volume element to a prior probability distribution ( $p_{F_{q=1}}$ ) over the Bloch sphere, we must divide by 0.245 592 93.

### 5. Comparative noninformativity analysis

We have that  $S_{\text{KL}}(p_F \| p_{F_{q=1}}) = 0.229\,666$  and  $S_{\text{KL}}(p_{F_{q=1}} \| p_F) = 0.170\,145$ . Further, using the likelihood (16), based on six hypothetical measurements to generate posteriors, we obtain  $S_{\text{KL}}(\text{post}_F, p_{F_{q=1}}) = 0.707\,66$  and  $S_{\text{KL}}(\text{post}_{F_{q=1}} \| p_F) = 0.064\,1738$ . So, the comparative noninformativity test, which was initially developed by Clarke [10], leads us to a firm conclusion that the four-dimensional-based probability distribution  $p_{F_{q=1}}$  is *more* noninformative in nature than the three-dimensional-based  $p_F$ .

Additionally,  $S_{\text{KL}}(p_B \| p_{F_{q=1}}) = 0.148\,269$  and  $S_{\text{KL}}(p_{F_{q=1}} \| p_B) = 0.098\,9669$ . These are converted, respectively, to 0.283 218 and 0.084 287 if we replace the first arguments of the two relative entropy functionals by posterior distributions based on the (formal) square root ( $L(x, y, z)^{\frac{1}{2}}$ ) of the likelihood function (16). Thus, we can conclude that  $p_{F_{q=1}}$  is also *more* noninformative than  $p_B$ .

Further,  $S_{\text{KL}}(p_{B_{q=1}\text{trunc}} \| p_{F_{q=1}}) = 0.105\,463$  and  $S_{\text{KL}}(p_{F_{q=1}} \| p_{B_{q=1}\text{trunc}}) = 0.091\,4175$ . Again, using the formal square root ( $L(x, y, z)^{\frac{1}{2}}$ ) of the likelihood, we obtain changes, respectively, to 0.245 602 and 0.040 8236. So, our conclusion here is that  $p_{F_{q=1}}$  is also more noninformative than  $p_{B_{q=1}\text{trunc}}$ . We already know from [30] that  $p_B$  is considerably more noninformative than  $p_F$ .

Continuing along these lines,  $S_{\text{KL}}(p_{B_{q=1}\text{trunc}} \| p_F) = 0.019\,1948$  and  $S_{\text{KL}}(p_F \| p_{B_{q=1}\text{trunc}}) = 0.023\,4599$  (so the two distributions are relatively close to one another). Using ( $L(x, y, z)^{\frac{1}{2}}$ ) to generate posterior distributions, the first statistic is altered (slightly decreased) to 0.014 3147, while the second statistic jumps to 0.104 7772.

So, assembling these several relative entropy statistics, we have the previously indicated ordering of the four priors (1). (The conclusions of the comparative noninformativity test appear to be *transitive* in nature, although I can cite no explicit theorem to that effect.)

#### 5.1. Relation to Srednicki's criterion for priors

In figure 1, we show the one-dimensional marginal probabilities of the four prior probabilities over the radial coordinate  $r$  in the near-to-pure-state range  $r \in [0.995, 1]$ . The dominance ordering in this plot *fully* complies with that (1) found by the information-theoretic-based comparative noninformativity test. (We note that this ordering is *not* simply *reversed* near the fully mixed state [ $r = 0$ ].) Conjecturally, this could be seen as a specific case of some (yet unproven) theorem—perhaps utilizing the convexity and decreasing-under-positive-mappings properties [34, p 35] of the relative entropy functional.

So, the information-theoretic (comparative-noninformativity) test appears to incorporate Srednicki’s criterion of ‘biasedness to pure states’ [9]. (Of course, it would be interesting to test the consistency between the comparative noninformativity test and Srednicki’s criterion with a larger number of priors, as well as in higher-dimensional quantum settings (cf [35]).) Srednicki does not explicitly observe that increasing biasedness to pure states corresponds to increasing noninformativity. He asserts that ‘we must decide how biased we are towards pure states’.

Srednicki focused on *two* possible priors. One was the uniform distribution over the Bloch sphere (unit ball). In [11, section 2.2], we had concluded that this distribution was *less* noninformative than  $p_B$ , in full agreement with contemporaneous work of Hall [13]. The second prior (‘the Feynman measure’), which Srednicki points out is less biased to the pure states than the uniform distribution, was discussed in [36]. Neither of the two priors analysed by Srednicki corresponds to the normalized volume element of a *monotone* metric [11, 36].

**6.  $q$ -extended inference**

In the setting of the  $q$ -parameterized escort density matrices (9), the factor  $\frac{1-z^2}{4}$  in the likelihood (16), giving the probability (in the standard three-dimensional Bloch sphere setting) of one spin-up and one spin-down being measured in the  $z$ -direction, would be *replaced* by

$$L_q(z) = \frac{r^2(1 + W^q)^2 - (-1 + W^q)^2 z^2}{4r^2(1 + W^q)^2}, \tag{22}$$

and similarly for the  $x$ - and  $y$ -directions. (For  $q = 1$ , we recover  $\frac{1-z^2}{4}$ .)

It would be interesting to ascertain if the volume elements of the extended four-dimensional (truncated) Bures and Husimi metrics ((10) and (21)) could be integrated over the product of the Bloch sphere *and*  $q \in [\frac{1}{2}, \infty]$  and normalized to (prior) probability distributions. Then, using likelihoods incorporating the form (22), one could conduct the comparative noninformativity test in a *four*-dimensional setting, rather than only the *three*-dimensional one employed throughout this study. It turns out, however, that the three-fold integral—holding  $q$  fixed—of the truncated volume element of (10) over the Bloch sphere is given by our formula (12). Therefore, the four-fold integral of the one-dimensional marginal over the indicated product region with  $q \in [\frac{1}{2}, \infty]$  must *diverge*. So, to achieve a *proper* probability distribution one would have to truncate  $q$  above a certain value.

Continuing along these lines, we omitted  $q$  above 500 (and below  $q = \frac{1}{2}$ ) and normalized the volume element of the (truncated) extended Bures metric to a proper probability distribution. Then, the information gain with respect to such a prior, using  $L_q(z)$ , is 0.059 7923 nats of information, while a *single* up or down measurement yields 0.134 651 nats, and two measurements along the same axis giving the same outcome leads to an information gain of 0.349 601. The analogous three (slightly *larger*) statistics, working in the unextended framework (where  $q$  does not explicitly enter, and is implicitly understood to equal 1), using  $p_B$  as prior, are, respectively,  $\frac{7}{6} - \log 3 \approx 0.068\ 0544$ , and

$$\frac{8_p F_q(\{\frac{1}{2}, 1, 2\}, \{\frac{3}{2}, \frac{5}{2}\}, 1) - \pi(-5 + \log(64)) - 6 - 12K}{6\pi} \approx 0.140\ 186, \tag{23}$$

(where  $_p F_q$  denotes a generalized hypergeometric function and  $K \approx 0.915\ 965\ 594\ 177$  is Catalan’s constant) and  $\frac{59}{30} - \log 5 \approx 0.357\ 229$ . (We encountered numerical difficulties using Mathematica in attempting to extend these analyses to measurements conducted in more than one direction, unless we restricted  $q$  to a range no larger than on the order of 10.)

One might also consider the possible relevance of  $q$ -analogues of the Clarke comparative noninformativity test, using  $q$ -relative entropy (Kullback–Leibler) divergence [37, 38].

### 7. $q$ -extended Bures metric for higher-dimensional quantum scenarios

#### 7.1. Four-variable $3 \times 3$ density matrices

In [35], we considered an extension of the  $2 \times 2$  density matrices (4) to the  $3 \times 3$  form (by incorporating an additional parameter  $v$ )

$$\rho = \frac{1}{2} \begin{pmatrix} v+z & 0 & x-iy \\ 0 & 2-2v & 0 \\ x+iy & 0 & v-z \end{pmatrix}, \quad r^2 = x^2 + y^2 + z^2 \leq v^2, \quad 0 \leq v \leq 1. \quad (24)$$

The Bures metric was found there to take the form

$$d_{B_{n=3}}(\rho, \rho + d\rho)^2 = \frac{1}{4} \left( \frac{r^2 - v}{(1-v)(r^2 - v^2)} dv^2 + \frac{r}{r^2 - v^2} dv dr + \frac{v}{v^2 - r^2} dr^2 + \frac{1}{v} dn^2 \right). \quad (25)$$

(So, the tangential component is *independent* of  $r$ , as with (6) (cf [13]).) Normalizing the volume element of (25), we obtain the prior probability distribution [35, equation (18)]

$$p_{B_{n=3}} = \frac{3r^2 \sin \theta_1}{4\pi^2 v \sqrt{1-v} \sqrt{v^2 - r^2}}. \quad (26)$$

We have calculated that the (five-dimensional)  $q$ -extension of this metric has a tangential component of the form

$$\frac{((-r+v)^q - (r+v)^q)^2}{4r^2((-r+v)^q + (r+v)^q)((2-2v)^q + (-r+v)^q + (r+v)^q)}, \quad (27)$$

but have not yet been able to derive simple forms for the other entries of this metric tensor.

Numerical tests appear to indicate that the volume element of this  $q$ -extended Bures metric tensor is (also) identically zero.

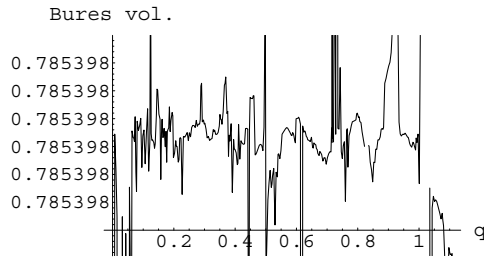
#### 7.2. Abe–Rajagopal two-qubit states

Since our first two attempts above to extend the Bures metric from an  $n$ -dimensional setting to an  $(n + 1)$ -dimensional framework, by embedding the  $q$ -order parameter, have yielded metrics (one of them being (10)) with zero volume elements, we were curious as to whether or not we could obtain, in some other quantum context, a *nondegenerate*  $q$ -extension of the Bures metric. In this regard, we turned our attention to the paper, ‘Quantum entanglement inferred by the principle of maximum nonadditive entropy’ of Abe and Rajagopal [7] (cf [8, equation (14)]).

Their principal object of study is a  $4 \times 4$  density matrix [7, equation (32)], being ostensibly parameterized by *three* variables, the order (nonadditivity) parameter  $q$ , the  $q$ -expected value  $b_q$  of the Bell–CHSH observable and its dispersion  $\sigma_q^2$ . (Two of the four eigenvalues of the density matrix are always equal.)

We applied the Hübner formula (3) for the Bures metric to this family of  $4 \times 4$  density matrices, considering  $q$  as a freely-varying parameter, along with  $b_q$  and  $\sigma_q^2$ . Computing the  $3 \times 3$  Bures metric tensor, and *then* setting  $q = 1$ , we obtain the metric

$$ds_{AbeRaj_{q=1}}(\rho, \rho + d\rho)^2 = \frac{c}{1024} dq^2 + \frac{\log(-2\sqrt{2}b_q + \sigma_q^2) - \log(2\sqrt{2}b_q + \sigma_q^2)}{8\sqrt{2}} dq db_q + \frac{2 \log(8 - \sigma_q^2) - \log(-2\sqrt{2}b_q + \sigma_q^2) - \log(2\sqrt{2}b_q + \sigma_q^2)}{32} dq d\sigma_q^2$$



**Figure 9.** Numerical integration estimates of the Bures volume of the (separable and nonseparable) AR two-qubit states, as a function of  $q$ . This volume is known to be  $\frac{\pi}{4} \approx 0.7853981634$  for  $q = \frac{1}{2}$  and 1 and appears to be so for all (positive)  $q$ .

$$\begin{aligned}
 & + \frac{\sigma_q^2}{-32b_q^2 + 4(\sigma_q^2)^2} (db_q)^2 + \frac{b_q}{16b_q^2 - 2(\sigma_q^2)^2} db_q d\sigma_q^2 \\
 & + \frac{b_q^2 - \sigma_q^2}{4(-8 + \sigma_q^2)(-8b_q^2 + (\sigma_q^2)^2)} (d\sigma_q^2)^2.
 \end{aligned} \tag{28}$$

Here, we have

$$\begin{aligned}
 c = & -4 \log(8 - \sigma_q^2)^2 \sigma_q^2 (-8 + \sigma_q^2) + 2 \log(-2\sqrt{2}b_q + \sigma_q^2) \log(2\sqrt{2}b_q + \sigma_q^2) (8b_q^2 - \sigma_q^4) \\
 & - \log(-2\sqrt{2}b_q + \sigma_q^2)^2 (8b_q^2 + \sigma_q^2 (-16 + \sigma_q^2) - 4\sqrt{2}b_q (-8 + \sigma_q^2)) \\
 & - \log(2\sqrt{2}b_q + \sigma_q^2)^2 (8b_q^2 + \sigma_q^2 (-16 + \sigma_q^2) + 4\sqrt{2}b_q (-8 + \sigma_q^2)) \\
 & + 4 \log(8 - \sigma_q^2) (-8 + \sigma_q^2) (\log(-2\sqrt{2}b_q + \sigma_q^2) (-2\sqrt{2}b_q + \sigma_q^2) \\
 & + \log(2\sqrt{2}b_q + \sigma_q^2) (2\sqrt{2}b_q + \sigma_q^2)).
 \end{aligned} \tag{29}$$

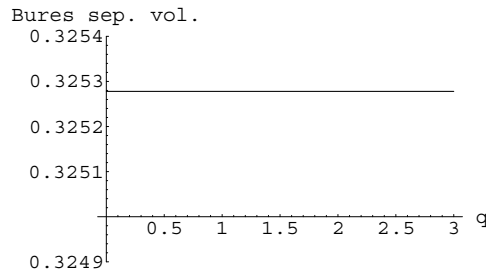
Numerical computations indicate that the volume element of the metric  $ds_{\text{AbeRaj}_q}(\rho, \rho + d\rho)^2$ , for any value of  $q$ , is zero.

In the *unextended* (two-parameter) case, the *nondegenerate* volume element (with  $q = 1$ ) is

$$dV_{\text{AbeRaj}_{q=1}} = \frac{\sqrt{-\left(\frac{1}{(-8 + \sigma_q)(-8b_q^2 + (\sigma_q^2)^2)}\right)}}{4} db_q d\sigma_q^2. \tag{30}$$

In [39], it was asserted that for the cases  $q = \frac{1}{2}$  and 1, the associated *separability probabilities* of the Abe–Rajagopal (AR) states were equal to the ‘silver mean’, that is,  $\sqrt{2} - 1 \approx 0.414214$  (cf [40, 41]). We have reconfirmed these two *probabilities*, while also finding that the Bures volume of separable and nonseparable states is, in both these cases, equal to  $\frac{\pi}{4} \approx 0.7853981634$ . This also appears to be the Bures volume for all positive  $q$ , as indicated by the results obtained by numerical integration presented in figure 9. The integrand employed (that is, the Bures volume element) was

$$dV_{\text{AbeRaj}} = 16 \sqrt{\frac{(8 - \sigma_q)^{\frac{1}{q}-2} (\sigma_q - 2\sqrt{2}b_q)^{\frac{1}{q}} (2\sqrt{2}b_q + \sigma_q)^{\frac{1}{q}}}{q^4 (\sigma_q^2 - 8b_q^2)^2 (2(8 - \sigma_q)^{\frac{1}{q}} + (\sigma_q - 2\sqrt{2}b_q)^{\frac{1}{q}} + (2\sqrt{2}b_q + \sigma_q)^{\frac{1}{q}})^3}} db_q d\sigma_q^2. \tag{31}$$



**Figure 10.** Numerical integration estimates of the Bures volume of the separable A two-qubit states, as a function of  $q$ . This volume is known to be exactly  $\frac{\pi(\sqrt{2}-1)}{4} \approx 0.325\ 323$  for  $q = \frac{1}{2}$  and 1 and appears to be so for all (positive)  $q$ .

It also appears (figure 10) that the Bures volume of the separable (only) AR-states is equal to  $\frac{\pi(\sqrt{2}-1)}{4} \approx 0.325\ 323$  for all positive  $q$  and, thus, the separability probabilities (obtained by taking the ratios) are all simply  $\sqrt{2} - 1 \approx 0.414\ 214$  (that is, the ‘silver mean’). (The numerical integration employed to generate figure 10 is more challenging—due to the necessary imposition of the Peres separability criterion—than to create figure 9, so we could not obtain as many significant digits.)

These  $q$ -invariance results stand in interesting contrast to the emphasis of Abe and Rajagopal ‘that for  $q > 1$ , indicating the subadditive feature of the Tsallis entropy, the entangled region is small and enlarges as one goes into the superadditive regime where  $q < 1$ ’ [7, p 3464 and figure 1]. But, in terms of the Bures metric (and others we will see below) the *measure* of the region does *not* change with  $q$ .

Using the *Hilbert–Schmidt* metric [42], rather than the Bures, we find that the volume of separable and nonseparable AR two-qubit states is equal to  $\frac{1}{4\sqrt{2}}$  for both  $q = \frac{1}{2}$  and 1 and the volume of separable states is equal to  $\frac{1}{8\sqrt{2}}$  for both these values of  $q$ , so the corresponding Hilbert–Schmidt separability probabilities are simply  $\frac{1}{2}$ . (If we employ either the *Wigner–Yanase* (monotone) metric [43] or the *arithmetic average* (monotone) metric [41], then, for  $q = 1$ , we obtain exactly the same (volume) results as using the Bures metric, and for  $q = \frac{1}{2}$ —using numerical rather than symbolic methods in the Wigner–Yanase case—quite clearly the same also.) So, it certainly appears that the  $q$ -invariance of the total and separable volumes of the AR-states is metric *independent*. Canosa and Rossignoli [44, p 4] have noted that for the AR-states, the ‘final maximum entropy density is actually *independent of the choice of  $f$* ’, where  $f$  is a smooth *concave* function.

The Bures separability probability (as well as that based on the Wigner–Yanase metric) of the (one-parameter) ‘Jaynes state’ [3, 1], in which (unlike the AR-states) no constraint on the dispersion is present (and  $q$  is implicitly equal to 1), is  $\frac{2 \arcsin(\sqrt{2}-1)}{\pi} \approx 0.271\ 887$ . (Again, note the presence of the silver mean—and implicitly in the very next formula.) The Hilbert–Schmidt separability probability is

$$\frac{\sinh^{-1}(2 - \sqrt{2}) + \text{Root}[\#1^4 - 148\#1^2 + 68\&, 3]}{\sqrt{6} + \sinh^{-1}(\sqrt{2})} \approx 0.343\ 602. \tag{32}$$

To convert from the AR two-qubit density matrix for  $q = 1$  to that for  $q = \frac{1}{2}$ , we merely have to perform the transformation

$$\left\{ \sigma_1^2 \rightarrow \frac{4(8b_{\frac{1}{2}}^2 + (\sigma_{\frac{1}{2}}^2)^2)}{4b_{\frac{1}{2}}^2 + (\sigma_{\frac{1}{2}}^2)^2 - 8\sigma_{\frac{1}{2}}^2 + 32}, b_1 \rightarrow \frac{8b_{\frac{1}{2}}\sigma_{\frac{1}{2}}^2}{4b_{\frac{1}{2}}^2 + (\sigma_{\frac{1}{2}}^2)^2 - 8\sigma_{\frac{1}{2}}^2 + 32} \right\}. \quad (33)$$

Presumably, there is a (more complicated, in general) transformation between AR-states for any pair of distinct values of  $q$ . So, in retrospect, the  $q$ -invariance of the (Bures, Hilbert–Schmidt, Wigner–Yanase and arithmetic average) metric volumes is not so surprising, since we are simply working within one family of *two*-parameter density matrices, the various  $q$ -manifestations of which can be obtained by suitable reparameterizations. Similarly, the null nature of the  $q$ -extended Bures metric for the AR-states can be seen in this light.

7.2.1. *Trivariate Jaynes state using generalized Bell–CHSH observables.* It would be interesting to extend and analyse the AR-states based on modifications of the Bell–CHSH observable (cf [2, section 3] [45]). In fact, we pursued such a line of investigation, using

$$B_\alpha = 2\sqrt{2}(|\Phi^+\rangle\langle\Phi^+| - \alpha|\Psi^-\rangle\langle\Psi^-|), \quad (34)$$

as the observable, where for  $\alpha = 1$ , we recover the Bell–CHSH observable employed by Abe and Rajagopal [7, equation (6)] (cf [2, equation (15)]). (We utilized the Jaynes maximum entropy strategy [3], implicitly taking  $q = 1$ —so, most precisely, we are extending the model discussed by Rajagopal in [1] to incorporate generalized Bell–CHSH observables or, alternatively, the Canosa–Rossignoli model to include the dispersion.) Then, the volume element of the Bures metric, considering  $\alpha$  as a parameter, in addition to the expectation  $b_1$  and the dispersion  $\sigma_1$ , was *null*. Considering, on the other hand,  $\alpha$  to be simply a fixed constant, the bivariate Bures volume element was of the (non-null) form  $\frac{1}{4}\sqrt{\frac{1}{C+D}} db_1 d\sigma_1^2$ , where

$$C = (8\alpha - \sigma_1^2)(\sigma_1^2)^2 - 4\sqrt{2}(\alpha - 1)b_1(\sigma_1^2 - 4\alpha)\sigma_1^2 \quad (35)$$

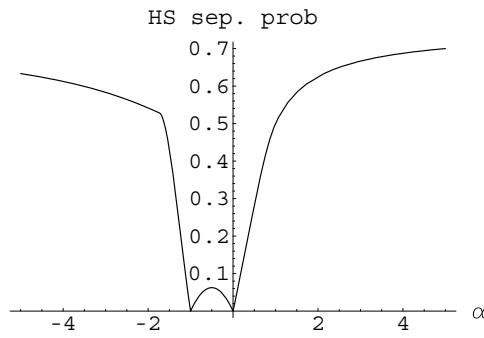
and

$$D = 16\sqrt{2}(\alpha - 1)\alpha b_1^3 - 8b_1^2((\sigma_1^2 + 8)\alpha^2 - 3\sigma_1^2\alpha + \sigma_1^2).$$

For  $\alpha = 1$ , we recover (30), so the associated Bures separability probability is the silver mean. For  $\alpha = 1$ , using the HS-metric now, the total volume of states is  $\frac{1}{4\sqrt{2}}$  and that of the separable states is  $\frac{1}{8\sqrt{2}}$ , so the HS separability probability is simply  $\frac{1}{2}$ .

For the case  $\alpha = 2$ , we obtained a result of 0.353 68 for the Bures volume of separable and nonseparable states, and 0.200 0322 for the Bures volume of only separable states, yielding a separability probability of 0.566 392. The comparable results for the Hilbert–Schmidt case (the volume element—*independent* of  $b_q$  and  $\sigma_q^2$ —being  $\frac{1}{32\alpha(1+\alpha)} db_q d\sigma_q^2$ ) were  $\frac{1}{12\sqrt{2}}$  and  $\frac{5}{96\sqrt{2}}$ , with a separability probability of  $\frac{5}{8}$ .

Exact integration, then, gave the HS volume of separable and nonseparable states to equal, *in general*,  $\frac{1}{2\sqrt{2}(\alpha+\alpha^2)}$ , and that of the separable states—but only for  $\alpha \geq 1$ —to be  $\frac{3\alpha-1}{8\sqrt{2}\alpha^2(1+\alpha)}$ , so the Hilbert–Schmidt separability probability for  $\alpha \geq 1$  is simply equal to  $\frac{3}{4} - \frac{1}{4\alpha}$ . For  $\alpha = \frac{1}{2}$ , the HS separability probability appeared to be  $\frac{37}{64} \approx 0.578\ 125$ .



**Figure 11.** Separability probabilities (37) based on the Hilbert–Schmidt metric, as a function of the parameter  $\alpha$ , for the new class of trivariate Jaynes states employing *generalized* Bell, Clauser, Horne, Shimony, Holt (Bell–CHSH) observables (34).

Then, using the integration over implicitly defined regions feature new to Mathematica 5.1, we were able to obtain the HS *separable* volumes, for all (real) values of  $\alpha$ ,

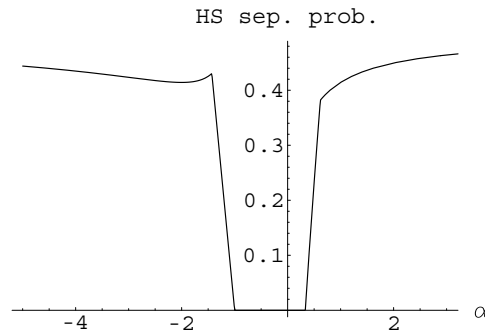
$$\left( \begin{array}{ll} -\frac{1}{8\sqrt{2}} & -1 < \alpha < 0 \\ \frac{5}{16\sqrt{2}} & \alpha = -\sqrt{2} \\ \frac{1}{32}(5\sqrt{2} - \sqrt{10}) & \alpha = -\frac{1}{2} + \frac{\sqrt{5}}{2} \\ \frac{1}{16}(2\sqrt{2} - \sqrt{2}\alpha) & 0 < \alpha < -\frac{1}{2} + \frac{\sqrt{5}}{2} \vee -\frac{1}{2} + \frac{\sqrt{5}}{2} < \alpha < 1 \\ \frac{3\sqrt{2}\alpha - \sqrt{2}}{16\alpha^2(\alpha+1)} & \alpha \geq 1 \\ \frac{3\sqrt{2}\alpha^2 + \sqrt{2}}{16\alpha^2(\alpha^2-1)} & \alpha \leq -\sqrt{3} \\ \frac{-\sqrt{2}\alpha^4 + 5\sqrt{2}\alpha^2 - \sqrt{2}}{16\alpha^2} & -\sqrt{2} < \alpha < -1 \vee -\sqrt{3} < \alpha < -\sqrt{2} \end{array} \right) \quad (36)$$

and, dividing by the total HS volume ( $\frac{1}{2\sqrt{2}(\alpha+\alpha^2)}$ ), the HS separability *probability* results,

$$\left\{ \begin{array}{ll} -\frac{1}{4}\alpha(\alpha+1) & -1 < \alpha < 0 \\ \frac{5}{8}\alpha(\alpha+1) & \alpha + \sqrt{2} = 0 \\ -\frac{1}{8}(-5 + \sqrt{5})\alpha(\alpha+1) & 2\alpha + 1 = \sqrt{5} \\ -\frac{1}{4}(\alpha - 2)\alpha(\alpha+1) & 0 < \alpha < \frac{1}{2}(-1 + \sqrt{5}) \vee \frac{1}{2}(-1 + \sqrt{5}) < \alpha < 1 \\ \frac{3}{4} - \frac{1}{4\alpha} & \alpha \geq 1 \\ -\frac{1}{4\alpha} + \frac{3}{4} + \frac{1}{\alpha - 1} & \alpha + \sqrt{3} \leq 0 \\ -\frac{(\alpha+1)(\alpha^4 - 5\alpha^2 + 1)}{4\alpha} & -\sqrt{2} < \alpha < -1 \vee -\sqrt{3} < \alpha < -\sqrt{2}. \end{array} \right. \quad (37)$$

In figure 11, we plot these rather interesting/intricate results. The separability probabilities are *zero* at the isolated points  $\alpha = -1, 0$ . For  $\alpha \rightarrow \pm\infty$ , a maximum of  $\frac{3}{4}$  is approached. We also see that the ‘golden ratio’ (or ‘golden mean’) [5, 6] (or its inverse, depending upon the





**Figure 12.** Separability probabilities (38) based on the Hilbert–Schmidt metric, as a function of the parameter  $\alpha$ , for the (Canosa–Rossignoli) class of *bivariate* Jaynes states employing *generalized* Bell, Clauser, Horne, Shimony, Holt (Bell–CHSH) observables (34). The variance  $\sigma_1^2$  has been set to its value  $4(1 + \frac{b_1^2}{8})$  in the single constraint (Horodecki) case.

definition),  $\frac{\sqrt{5}-1}{2}$ , enters into delineating the different segments over which the separability probabilities take different functional forms. (It would seem plausible, although we have not conducted a full, detailed analysis that the points at which the functional forms change, correspond to separability constraints passing from inactive to active roles, and vice versa.) *Fake* entanglement is avoided in this three-parameter model [2, p 128] [44, p 4]. It did not appear feasible to directly expand the *trivariate*  $(b_q, \sigma_q^2, \alpha)$  scenario just investigated to a *quadrivariate* one.

7.2.2. *Bivariate Jaynes state using generalized Bell–CHSH observables.* Let us, however, consider a related *bivariate* (Canosa–Rossignoli-type [2, p 126]) model, in which we set  $\sigma_1^2 = 4(1 + \frac{b_1^2}{8})$ , the dispersion in the single constraint case [1, equation (13)]. Then, the HS separability probabilities take the form (the HS volume here being  $\frac{2\sqrt{3\alpha^4 - 2\alpha^2 + 3}}{4\alpha^2 + 4\alpha}$ ),

$$\left\{ \begin{array}{ll} -\alpha - 1 & \frac{1}{3}(1 - 2\sqrt{7}) < \alpha < -1 \\ \sqrt{(\alpha - 2)\alpha} - \sqrt{\alpha(\alpha + 1)} - 1 & 3\alpha + 2\sqrt{7} \leq 1 \\ \sqrt{\alpha(\alpha + 1)} - \alpha & 2\alpha + 1 \geq \sqrt{5} \\ -\alpha + 2\sqrt{\alpha(\alpha + 1)} - 1 & \frac{1}{3} < \alpha < \frac{1}{2}(-1 + \sqrt{5}). \end{array} \right. \quad (38)$$

We represent this in figure 12. In the  $\pm\infty$  limit, the separability probability approaches  $\frac{1}{2}$ . The HS separability probability is zero in the interval  $\alpha \in [-1, \frac{1}{3}]$ . The point  $\alpha = 1$  corresponds to the use of the standard (ungeneralized) Bell–CHSH observable ( $B_1$ ), and to the *one*-constraint Horodecki model [3]. We can see that the associated HS separability probability is the silver mean.

7.3. *Tsallis–Lloyd–Baranger two-qubit states*

Tsallis, Lloyd and Baranger have considered a scenario in which the probabilities of being in either one of the four states of the Bell basis is given in the form  $\frac{(1-x)}{4}, \frac{(1-y)}{4}, \frac{(1-z)}{4}$  and  $\frac{(1+x+y+z)}{4}$  [8]. (The feasible points  $(x, y, z)$  lie in a certain tetrahedron [8, figure 3].) They also embed their three-parameter (two-qubit)  $4 \times 4$  density matrix [8, equation (12)] into an

(unnormalized) four-parameter  $4 \times 4$  density matrix [8, equation (14)] by introducing the  $q$ -parameter.

Upon its normalization and application of the Hübner formula (3), we obtained the corresponding Bures metric, the volume element of which, in a numerical investigation, appeared to be zero, in this four-parameter extended case. So, we have, to this point, yet to find any *nondegenerate*  $q$ -extension of the Bures metric (if one is so possible).

In the unextended three-parameter case, if we employ new coordinates of the form,

$$x = 1 - 4 \cos \theta_1, \quad y = 1 - 4 \sin \theta_1 \cos \theta_2, \quad z = 1 - 4 \sin \theta_1 \sin \theta_2 \cos \theta_3, \quad (39)$$

then we have simply

$$d_{TLB}(\rho, \rho + d\rho)^2 = d\theta_1^2 + \sin^2 \theta_1 d\theta_2^2 + \sin^2 \theta_1 \sin^2 \theta_2 d\theta_3^2, \quad (40)$$

that is, the *uniform* metric on the 3-sphere.

The Bures *volume* of the (separable and nonseparable) TLB-states is  $\frac{\pi^2}{8}$ , while the Bures volume of just the separable states themselves is (thanks to a challenging computation—involving a *cylindrical algebraic decomposition* [cad] [46]—performed by M. Trott)  $\frac{\pi(4-\pi)}{8}$ . (The separable states comprise the cube  $x, y, z \in [-1, \frac{1}{3}]$ .) Thus, the Bures separability probability [39–41] of the TLB-states is (quite elegantly)  $\frac{4-\pi}{\pi} \approx 0.273\ 24$ .

For the Hilbert–Schmidt metric, the volume of the TLB (separable and nonseparable) states is  $\frac{1}{6\sqrt{2}}$  and that of the separable states,  $\frac{1}{12\sqrt{2}}$ , so the HS separability probability is  $\frac{1}{2}$ .

### 8. Rajagopal–Abe metric

We also investigated the possible application of the ‘generalized metric’ (based on the  $q$ -Kullback–Leibler entropy) [47, equation (16)] to the two-level quantum systems (4)—and seeing how it pertains to the family of quantum monotone metrics [16]. For the case  $q = 1$  (which should reduce to the Kullback–Leibler symmetrized divergence [47, equation (6)], our calculations yielded that the diagonal elements of the ‘metric’ take the form

$$\begin{aligned} & \frac{1}{2} \left( -\log \left( -\frac{r}{2+r} \right) + \log W \right) dr^2 - \frac{1}{4} r (3 + \cos 2\theta_1 + 2 \sin 2\theta_1) \log W d\theta_1^2 \\ & - \frac{1}{4} r (3 + \cos 2\theta_2 + 2 \sin 2\theta_2) \log W \sin^2 \theta_1 d\theta_2^2, \end{aligned} \quad (41)$$

where, as we recall,  $W = \frac{1-r}{1+r}$ . So, it clearly cannot possess the form required of a quantum monotone metric (cf section 3.1). However, when we attempted to implement equation (6) of ([47]), bypassing the  $q$ -framework, we obtained for the (presumably same?) metric

$$\frac{1}{1-r^2} dr^2 - \frac{1}{2r} \log W dn^2, \quad (42)$$

which does not appear to correspond to a *monotone* metric.

### 9. Concluding remarks

Naudts [22] introduced the concept of a  $\phi$ -exponential family of density operators  $\rho_\theta$  (for which the obvious example is  $\phi(u) = u^q$ ). He showed that the  $\phi$ -exponential family of density operators, together with a family of escort density operators, optimizes a generalized version of the well-known Cramér–Rao lower bound. He assumes that certain Hamiltonians are two-by-two commuting. Therefore, the quantum information manifold  $(\rho_\theta)_\theta$  is abelian, which ‘is clearly too restrictive for a fully quantum-mechanical theory’. He suggests further work to remove this restriction.

Abe regarded the order of the escort distribution  $q$  as a parameter [48]. He studied the geometric structure of the one-parameter family of escort distributions using the Kullback divergence, and showed that the Fisher metric is given in terms of the generalized bit variance, which measures fluctuations of the crowding index of a multifractal.

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